# Linear systems – Final exam – Solutions

Final exam 2024–2025, Tuesday 17 June 2025, 15:00 – 17:00

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**Instructions**

1. The use of books, lecture notes, or (your own) notes is not allowed.
2. All answers need to be accompanied with an explanation or calculation.

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## Problem 1

$(8 + 4 + 6 + 12 + 4 = 34$ points$)$

Consider the model

$$\begin{aligned} A_1\dot{h}_1(t) &= -a_1\sqrt{2gh_1(t)} + u(t), \\ A_2\dot{h}_2(t) &= -a_2\sqrt{2gh_2(t)} + a_1\sqrt{2gh_1(t)}, \end{aligned} \tag{1}$$

describing the water height in two interconnected tanks. Particularly, for tank $i \in \{1, 2\}$, $h_i(t) \in \mathbb{R}$ is the water height, $A_i > 0$ its cross-sectional area, and $a_i > 0$ the cross-sectional area of the outlet. The input $u(t) \in \mathbb{R}$ represents the inflow to the first tank, and $g$ denotes the graviational constant.

(a) Consider initial conditions $h_i(0) = h_{i,0}$ and assume that $h_{i,0} \geq 0$ for $i \in \{1, 2\}$. Moreover, assume that the inflow is constant and nonnegative, i.e., $u(t) = \bar{u}$ for all $t \geq 0$, with $\bar{u} \geq 0$. On the basis of the dynamics (1), explain that $h_1(t) \geq 0$ and $h_2(t) \geq 0$ for all $t \geq 0$.

(b) In the notation of (a), take $h_{i,0} > 0$ for $i \in \{1, 2\}$ and $\bar{u} = 0$. The total water volume in the two tanks reads

$$V(h_1, h_2) = A_1h_1 + A_2h_2.$$

Show that the total water volume cannot increase as a function of time.

(c) Take $u(t) = \bar{u}$ for all $t$, with $\bar{u} > 0$. Find an equilibrium point $(\bar{h}_1, \bar{h}_2)$ and show that this equilibrium point is unique.

(d) Linearize the system (1) around the equilibrium point

$$\left( \begin{bmatrix} \bar{h}_1 \\ \bar{h}_2 \end{bmatrix}, \bar{u} \right).$$

*Hint.* In the linearized dynamics, simply write $\bar{h}_i$ rather than the expressions obtained in (c).

(e) Is the linearized system obtained in (d) asymptotically stable?

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### Answer Problem 1(a)

We first consider the first equation in (1). To explain that $h_1(t) \geq 0$, assume that there exists a time $t^*$ such that $h_1(t^*) = 0$. Then, the dynamics gives

$$A_1\dot{h}_1(t^*) = -a_1\sqrt{2gh_1(t^*)} + u(t^*) = \bar{u} \geq 0, \tag{2}$$

showing that $h_1$ cannot decrease below 0. Here, we note that $h_1(0) \geq 0$ and that the solution $h_1(\cdot)$ to the first equation of (1) is continuous.

Next, knowing that $h_1(t) \geq 0$, we follow the same reasoning on the second equation. For $t^*$ such that $h_2(t^*) = 0$, the dynamics reads

$$A_2\dot{h}_2(t^*) = -a_2\sqrt{2gh_2(t^*)} + a_1\sqrt{2gh_1(t^*)} = a_1\sqrt{2gh_1(t^*)} \geq 0, \tag{3}$$

such that $h_2$ cannot decrease below 0. Note that the $t^*$ in (3) is not necessarily the same as that in (2).

## Answer Problem 1(b)

Evulation of the volume $V$ along trajectories of (1) gives

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}V\big(h_1(t),h_2(t)\big) = A_1\dot{h}_1(t)+A_2\dot{h}_2(t) &= -a_1\sqrt{2gh_1(t)}-a_2\sqrt{2gh_2(t)}+a_1\sqrt{2gh_1(t)}\\ &= -a_2\sqrt{2gh_2(t)},\end{aligned}\tag{4}$$

which is nonpositive under the condition that $h_2(t)\geq 0$ for all $t\geq 0$. Hence, the total water volume cannot increase as a function of time.

## Answer Problem 1(c)

The equilibrium point $(\bar{h}_1,\bar{h}_2)$ satisfies the equations

$$\begin{aligned}0 &= -a_1\sqrt{2g\bar{h}_1}+\bar{u},\\ 0 &= -a_2\sqrt{2g\bar{h}_2}+a_1\sqrt{2g\bar{h}_1}.\end{aligned}\tag{5}$$

As $\bar{u}>0$ by assumption, the first equation has the unique solution

$$\bar{h}_1 = \frac{1}{2g}\left(\frac{\bar{u}}{a_1}\right)^2.\tag{6}$$

Summing the two equations gives

$$0 = -a_2\sqrt{2g\bar{h}_2}+\bar{u},\tag{7}$$

leading to the unique solution for $\bar{h}_2$ as

$$\bar{h}_2 = \frac{1}{2g}\left(\frac{\bar{u}}{a_2}\right)^2.\tag{8}$$

## Answer Problem 1(d)

As a first step in obtaining the linearization, denote

$$h = \begin{bmatrix}h_1\\ h_2\end{bmatrix},\qquad f(h,u) = \begin{bmatrix}-\frac{a_1}{A_1}\sqrt{2gh_1}+\frac{1}{A_1}u\\ -\frac{a_2}{A_2}\sqrt{2gh_2}+\frac{a_1}{A_2}\sqrt{2gh_1}\end{bmatrix}.\tag{9}$$

such that the dynamics (1) can be written compactly as

$$\dot{h}(t) = f\big(h(t),u(t)\big).\tag{10}$$

Next, we introduce the notation

$$\bar{h} = \begin{bmatrix}\bar{h}_1\\ \bar{h}_2\end{bmatrix}\tag{11}$$

and write the equilibrium $(\bar{h},\bar{u})$, which satisfies $0=f(\bar{h},\bar{u})$ by definition. We are interested in the deviations from the equilibrium, denoted by $\tilde{h}$ and $\tilde{u}$ and defined as

$$\tilde{h} = h-\bar{h},\qquad \tilde{u} = u-\bar{u}.\tag{12}$$

They satisfy the dynamics

$$\dot{\tilde{h}}(t) = \frac{\partial f}{\partial h}(\bar{h},\bar{u})\tilde{h}(t)+\frac{\partial f}{\partial u}(\bar{h},\bar{u})\tilde{u}(t).\tag{13}$$

Here, the Jacobian of $f$ with respect to $h$ is given by

$$\frac{\partial f}{\partial h}(h,u) = \begin{bmatrix} -\frac{a_1 g}{A_1\sqrt{2gh_1}} & 0 \\ \frac{a_1 g}{A_2\sqrt{2gh_1}} & -\frac{a_2 g}{A_2\sqrt{2gh_2}} \end{bmatrix}, \tag{14}$$

such that

$$\frac{\partial f}{\partial h}(\bar{h},\bar{u}) = \begin{bmatrix} -\frac{a_1 g}{A_1\sqrt{2g\bar{h}_1}} & 0 \\ \frac{a_1 g}{A_2\sqrt{2g\bar{h}_1}} & -\frac{a_2 g}{A_2\sqrt{2g\bar{h}_2}} \end{bmatrix}, \tag{15}$$

Next,

$$\frac{\partial f}{\partial u}(h,u) = \begin{bmatrix} \frac{1}{A_1} \\ 0 \end{bmatrix} = \frac{\partial f}{\partial u}(\bar{h},\bar{u}). \tag{16}$$

### Answer Problem 1(e)

Stability is determined by the eigenvalues of the system matrix (15). Due to the lower triangular structure of this matrix, it is clear that its spectrum simply comprises the diagonal elements, i.e.,

$$\sigma\left(\frac{\partial f}{\partial h}(\bar{h},\bar{u})\right) = \left\{-\frac{a_1 g}{A_1\sqrt{2g\bar{h}_1}}, -\frac{a_2 g}{A_2\sqrt{2g\bar{h}_2}}\right\}. \tag{17}$$

Clearly, both eigenvalues are real and negative (note that $\bar{h}_1 > 0$ and $\bar{h}_2 > 0$, see (6) and (8)), meaning that the linearization is asymptotically stable.

## Problem 2

(4 + 10 + 10 + 4 = 28 points)

Consider the linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \qquad \text{with} \qquad A = \begin{bmatrix} -5 & 17 \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 3 \end{bmatrix}.$$

and where $x(t) \in \mathbb{R}^2$, $u(t) \in \mathbb{R}$, and $y(t) \in \mathbb{R}$.

(a) Verify that the system is observable.

(b) Find a nonsingular matrix $T$ and real numbers $\alpha_1, \alpha_2$ such that

$$TAT^{-1} = \begin{bmatrix} 0 & \alpha_1 \\ 1 & \alpha_2 \end{bmatrix}, \quad CT^{-1} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

(c) Use the matrix $T$ from (b) to design a stable state observer of the form

$$\dot{\xi}(t) = A\xi(t) + Bu(t) + G(y(t) - C\xi(t)).$$

In particular, ensure that $\sigma(A - GC) = \{-2, -3\}$.

(d) Can the system be stabilized by dynamic output feedback?

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### Answer Problem 2(a)

To verify observability, compute

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix}. \tag{18}$$

Hence, we have

$$\operatorname{rank} \begin{bmatrix} C \\ CA \end{bmatrix} = \operatorname{rank} \begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix} = 2, \tag{19}$$

which equals the state-space dimension. We conclude that the system is observable.

### Answer Problem 2(b)

As a first step in finding the matrix $T$, compute the characteristic polynomial of $A$ as

$$\Delta_A(s) = \det(sI - A) = \begin{vmatrix} s+5 & -17 \\ 1 & s-4 \end{vmatrix} = (s+5)(s-4) + 17 = s^2 + s - 3. \tag{20}$$

This is of the form

$$\Delta_A(s) = s^2 + a_1 s + a_0, \tag{21}$$

with

$$a_1 = 1, \qquad a_0 = -3. \tag{22}$$

By the observability canonical form, there exists a nonsingular matrix $T \in \mathbb{R}^{2\times 2}$ that achieves the desired form, with

$$\alpha_1 = -a_0 = 3, \qquad \alpha_2 = -a_1 = -1. \tag{23}$$

To explicitly construct this $T$, consider

$$q_2 = C = \begin{bmatrix} -1 & 3 \end{bmatrix} \tag{24}$$

and

$$q_1 = CA + a_1 C = \begin{bmatrix} 2 & -5 \end{bmatrix} + 1 \begin{bmatrix} -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \end{bmatrix}, \tag{25}$$

after which $T$ is constructed as

$$T = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}. \tag{26}$$

**Answer Problem 2(c)**

First note that, for any nonsingular matrix $T$,

$$\Delta_{A-GC}(s) = \Delta_{T(A-GC)T^{-1}}(s). \tag{27}$$

In particular, choosing the matrix $T$ from Problem 2(b), we obtain

$$T(A-GC)T^{-1} = TAT - TGCT^{-1} = \begin{bmatrix} 0 & -a_0 \\ 1 & -a_1 \end{bmatrix} - \begin{bmatrix} g_0 \\ g_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}, = \begin{bmatrix} 0 & -a_0 - g_0 \\ 1 & -a_1 - g_1 \end{bmatrix} \tag{28}$$

where we have defined

$$TG = \begin{bmatrix} g_0 \\ g_1 \end{bmatrix}. \tag{29}$$

Exploiting the companion form of (28), we see that

$$\Delta_{T(A-GC)T^{-1}}(s) = s^2 + (a_1 + g_1)s + (a_0 + g_0). \tag{30}$$

As we would like to achieve $\sigma(A_G C) = \{-2, -3\}$, we introduce the desired polynomial

$$p(s) = (s+2)(s+3) = s^2 + 5s + 6, \tag{31}$$

which is of the form

$$p(s) = s^2 + p_1 s + p_0 \tag{32}$$

with

$$p_1 = 5, \qquad p_0 = 6. \tag{33}$$

Equating (30) and (32) leads to

$$g_1 = p_1 - a_1 = 5 - 1 = 4, \qquad g_0 = p_0 - a_0 = 6 - (-3) = 9, \tag{34}$$

where we have used (22) and (33). Recalling (29), we have

$$TG = \begin{bmatrix} 9 \\ 4 \end{bmatrix}, \tag{35}$$

after which the use of $T$ in (26) shows that

$$G = \begin{bmatrix} 35 \\ 13 \end{bmatrix}. \tag{36}$$

To obtain the latter step, we have used

$$T^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}. \tag{37}$$

### Answer Problem 2(d)

The system can be stabilized by dynamic output feedback if and only if $(A, B)$ is stabilizable and $(A, C)$ is detectable. We have verified in Problem 2(a) that $(A, C)$ is observable, which implies detectability. Hence, stabilizability of $(A, B)$ remains to be evaluated.

However, we simply evaluate controllability of $(A, B)$, which is sufficient for stabilizability. By direct computation,

$$\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ -1 & 4 \end{bmatrix}, \tag{38}$$

such that

$$\operatorname{rank} \begin{bmatrix} B & AB \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & -5 \\ -1 & 4 \end{bmatrix} = 2. \tag{39}$$

As this equals the state-space dimension, $(A, B)$ is controllable (and, hence, stabilizable).

We conclude that the system can be stabilized by dynamic output feedback.

## Problem 3 (12 points)

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Consider matrices $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $A$ such that $\lambda$ is not $(A, C)$-observable. Show that $\lambda$ is an eigenvalue of $A - GC$ for any $G \in \mathbb{R}^{n\times p}$.

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As $\lambda \in \sigma(A)$ is not $(A, C)$-observable, we have by definition that

$$\operatorname{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} \leq n. \tag{40}$$

In other words, there exists a nonzero $v \in \mathbb{C}^n$ such that

$$\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} v = 0. \tag{41}$$

Thus, $Av = \lambda v$ and $Cv = 0$. Now, pick any $G \in \mathbb{R}^{n\times p}$ and consider

$$(A - GC)v = Av - GCv = \lambda v - 0 = \lambda v, \tag{42}$$

i.e., $\lambda$ is an eigenvalue of $A - GC$.

## Problem 4 (16 points)

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Consider the two linear systems

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad \dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t),$$

with $x(t)$, $\bar{x}(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. Assume that both systems are controllable and, in addition, that they are similar, i.e., there exists a nonsingular $T \in \mathbb{R}^{n\times n}$ such that

$$\bar{A} = TAT^{-1}, \qquad \bar{B} = TB.$$

Show that $T$ is unique.

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Let $T \in \mathbb{R}^{n\times n}$ be a nonsingular matrix such that

$$\bar{A} = TAT^{-1}, \qquad \bar{B} = TB. \tag{43}$$

Then, we also have that

$$TA^kB = \bar{A}^k\bar{B} \tag{44}$$

for $k = 0, 1, 2, \ldots$, which we can easily prove by induction. Namely, for $k = 0$, the result simply follows from (43). Next, let (44) hold for some $k$ and consider

$$TA^{k+1}B = TAA^kB = \bar{A}TA^kB = \bar{A}\bar{A}^k\bar{B} = \bar{A}^{k+1}\bar{B}, \tag{45}$$

proving the inductive step. Here, we have used that $TA = \bar{A}T$, which is equivalent to $\bar{A} = TAT^{-1}$ in (43).

From (44), we now conclude that

$$T\begin{bmatrix}B & AB & \cdots & A^{n-1}B\end{bmatrix} = \begin{bmatrix}\bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{n-1}\bar{B}\end{bmatrix}. \tag{46}$$

To show that $T$ is unique, let $\bar{T}$ be another solution to (43). Then, following the same reasoning as above, we also have

$$\bar{T}\begin{bmatrix}B & AB & \cdots & A^{n-1}B\end{bmatrix} = \begin{bmatrix}\bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{n-1}\bar{B}\end{bmatrix}, \tag{47}$$

such that subtraction of (46) and (47) gives

$$(T - \bar{T})\begin{bmatrix}B & AB & \cdots & A^{n-1}B\end{bmatrix} = 0 \tag{48}$$

However, by controllability of the matrix pair $(A, B)$,

$$\operatorname{rank}\begin{bmatrix}B & AB & \cdots & A^{n-1}B\end{bmatrix} = n, \tag{49}$$

which implies $T = \bar{T}$.

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(10 points free)